

Testing the Dimension of Hilbert Spaces

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Given a set of correlations originating from measurements on a quantum state of unknown Hilbert space dimension, what is the minimal dimension d necessary to describe such correlations? We introduce the concept of dimension witness to put lower bounds on d . This work represents a first step in a broader research program aiming to characterize Hilbert space dimension in various contexts related to fundamental questions and quantum information applications.

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A theorist is invited to visit a lab. The experimentalists, not entirely happy with the nuisance, decide to submit the visitor to the ordeal “Guess what we are measuring.” Hardly distinguishing lasers from vacuum chambers, the theorist cannot hope to identify the system under study, and asks for a black-box description of the experiment in order to disentangle at least the physics from the cables. It turns out that the experiment aims at measuring correlations between the outputs of two measuring apparatuses. On each side, the outcome of the measurement is discrete and can take v values—the theorist writes $a, b \in \{0, 1, \dots, v-1\}$. A knob with m positions allows to change the parameters of each measuring apparatus—the theorist writes $x, y \in \{0, 1, \dots, m-1\}$. Finally, the experimentalists show the data: the frequencies $P(ab|xy)$ of occurrence of a given pair of outcomes for each pair of measurements. The theorist makes some calculations and delivers a verdict. . .

Some verdicts have been known for some time. In particular, if $P(ab|xy)$ violates a Bell-type inequality [1], we know for sure that an entangled quantum state has been produced in the lab. If on the contrary $P(ab|xy)$ can be distributed by shared randomness, the experiment may in fact be purely classical.

The goal of this Letter is to introduce another family of verdicts, different from the “quantum-vs-classical” one. We prove that, even in a black-box scenario, the theorist may have something to say about the dimension of the Hilbert space of the quantum objects that are measured. Both the enthusiastic verdict “You are using systems of dimension at least d ” and the disappointing one “You may be coding in less than d dimensions” are possible.

From an information-theoretical point of view, the dimensionality of quantum systems can be seen as a resource. Thus, testing the Hilbert space dimension is important for quantifying the power of quantum correlations, a central issue in Quantum Information science. Furthermore, this line of research turns out to be relevant for Quantum Key Distribution (QKD) as well. In standard

security proofs of QKD [2], the correlations shared by the authorized partners, Alice and Bob, are supposed to come from measurements on a quantum state of a given dimension. This assumption turns out to be crucial for the security of most of the existing protocols [3]. But is the dimension of a quantum system an experimentally measurable quantity? There also exist protocols whose security does not require any hypothesis on the Hilbert space dimension [4]. However, to prove security in such protocols, it is useful to understand how it is possible to bound effectively the dimension of the systems distributed by the eavesdropper [5].

Formally, a set of conditional probabilities $P(ab|xy)$ has a d -dimensional representation if it can be written as

$$P(ab|xy) = \text{tr}(\rho M_a^x \otimes M_b^y), \quad (1)$$

for some state ρ in $\mathbb{C}^d \otimes \mathbb{C}^d$ and local measurements operators M_a^x and M_b^y acting on \mathbb{C}^d , or if it can be written as a convex combination of probabilities of the form (1). We are interested in the following question: what is the minimal dimension d necessary to reproduce a given set of probabilities $P(ab|xy)$ [6]?

The fact that we allow convex combinations of (1) means that shared randomness is unrestricted in our scenario. This is consistent with a quantum information perspective where classical resources are taken to be free and we want to bound the quantum resources, in this case the dimensionality of the quantum states, necessary to achieve a task. Within this approach, the answer to the above question is immediate if the initial correlations admit a locally causal model [1], as in this case, they can be reproduced using shared randomness only and no quantum systems are strictly needed for their preparation. Thus, our problem is interesting only when the initial correlations are nonlocal.

Since classical correlations are taken to be free, the set of d -dimensional quantum correlations is convex. Therefore, standard techniques from convex theory can be applied, as has been done for other quantum information

problems such as separability [7]. Following this analogy, we introduce the concept of dimension witnesses. A d -dimensional witness is a linear function of the probabilities $P(ab|xy)$ described by a vector \vec{w} of real coefficients w_{abxy} , such that

$$\vec{w} \cdot \vec{p} \equiv \sum_{a,b,x,y} w_{abxy} P(ab|xy) \leq w_d \quad (2)$$

for all probabilities of the form (1) with ρ in $\mathbb{C}^d \otimes \mathbb{C}^d$, and such that there are quantum correlations for which $\vec{w} \cdot \vec{p} > w_d$. When some correlations violate (2), they can thus only be established by measuring systems of dimension larger than d . Dimension witnesses allow us to turn the Hilbert space dimension, a very abstract concept, into an experimentally measurable property.

In the following, we construct several examples of 2-dimensional witnesses. We also show that not all 2-outcome quantum correlations are achievable with qubits, answering a question raised by Gill [8]. A proof of the same result for two parties has been independently obtained in [9], while the results of [10] answer Gill's question in the tripartite case.

Witnesses based on CGLMP.—A natural starting point for our investigations is the situation corresponding to $m = 2$ measurement settings per side with $v = 3$ possible outcomes. Indeed, in this case Collins-Gisin-Linden-Massar-Popescu (CGLMP) introduced a Bell inequality whose maximal quantum violation is achieved by a two-qutrit state. The CGLMP expression is

$$C(\vec{p}) = P(b_0 \geq a_0) + P(a_0 \geq b_1) + P(a_1 \geq b_0) + P(b_1 > a_1) - 3 \quad (3)$$

where $P(a_x \geq b_y) = \sum_{a \geq b} P(ab|xy)$ [11,12]. Local correlations satisfy $C(\vec{p}) \leq 0$, while measurements on a partly entangled two-qutrit state yield a maximal value of $C(\vec{p}) = 0.3050$ [13].

The set of quantum probabilities corresponding to $m = 2$ and $v = 3$ lives in a 24-dimensional space. Since it is in general difficult to gain intuition in such a high-dimensional space, we will focus here on a two-dimensional subspace of this quantum set, which has been characterized in [14]. This subspace is parameterized by two numbers: the CGLMP value $C(\vec{p})$ and

$$D(\vec{p}) = - \sum_{x,y=0}^1 \sum_{k=0}^2 P[a=k, b=k-1-(x-1)(y-1)|xy].$$

The precise form of the probabilities living in this subspace as a function of $C(\vec{p})$ and $D(\vec{p})$ can be found in [14]. Note that if two parties share a point \vec{p} in the original quantum set, they can run a depolarization protocol that will map it onto the two-dimensional subspace while keeping the values of $C(\vec{p})$ and $D(\vec{p})$ constant [14].

Since the quantum region is convex, its boundary in the two-dimensional subspace can be obtained by computing the maximal value of

$$I_\phi(\vec{p}) = \cos\phi C(\vec{p}) + \sin\phi D(\vec{p}) \quad (4)$$

for all ϕ , that is by computing how far it extends in every direction of the two-dimensional subspace. We have computed these values using the technique introduced in [15]. The optimal values $I_\phi^{(q)}$ are obtained for entangled states of the form $|\psi\rangle = \frac{1}{\sqrt{2+\gamma^2}}(|00\rangle + \gamma|11\rangle + |22\rangle)$ with measurements that are independent of γ and which can be found in [11]. The resulting quantum curve is represented on Fig. 1 using the parametrization of [14].

We have also determined the region accessible with qubits by maximizing I_ϕ over all measurements (POVMs), and two-qubit states. The optimal values $I_\phi^{(2)}$ are obtained by performing two-outcome von Neumann measurements on pure entangled two-qubit states $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$. The resulting curve is also shown in Fig. 1. Note that contrarily to the previous case, the qubit curve is not the result of an exact computation, but of a numerical search using a heuristic algorithm. Indeed, the method of [15] cannot be directly applied here, as it does not constrain the dimension of the quantum systems, while others techniques based on semidefinite programming [16] are computationally too costly.

The inequalities $I_\phi(\vec{p}) \leq I_\phi^{(2)}$ form a family of 2-dimensional witnesses. Any one of these inequalities for which the maximal quantum value $I_\phi^{(q)} > I_\phi^{(2)}$ is strictly greater than the maximal qubit value, that is any direction in Fig. 1 for which there is a gap between the qubit and the general quantum curve, allows one to distinguish qubits

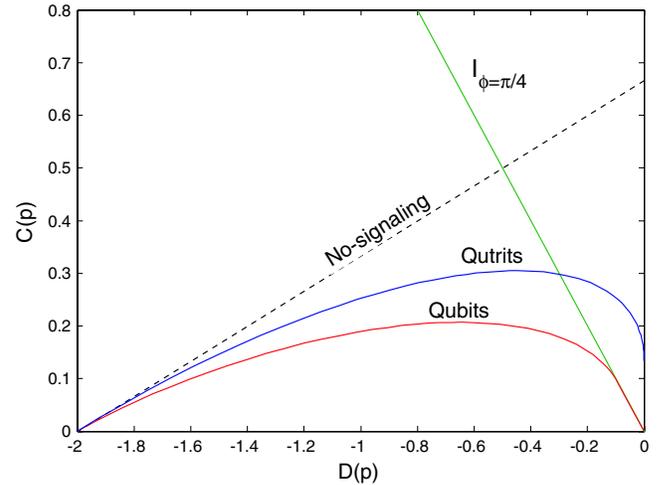


FIG. 1 (color online). Quantum region in the two-dimensional subspace described in [14]. The upper curve represents the boundary of the general quantum region and can be achieved by measurements on two-qutrit states. The lower curve represents the boundary of the region accessible through measurements on two-qubit states. The dashed line delimits the no-signaling correlations. The inequality $I_{\phi=\pi/4} \leq 0$ is a dimension witness: it cannot be violated by performing measurements on qubits, but qutrits are required.

from higher-dimensional systems. Note that the expressions (4) can also be interpreted as Bell inequalities with local bound $I_\phi^{(l)} = 0$ if $\sin\phi$ is positive and $I_\phi^{(l)} = -2\sin(\phi)$ otherwise. The inequalities with $\tan\phi \geq 1$ are noteworthy because the local bound and the qubit bound coincide, $I_\phi^{(2)} = I_\phi^{(l)} = 0$; i.e., qubit no longer violates them; they can only be violated with qutrits or higher-dimensional systems.

Although the situation that we just considered is illustrative, we mentioned that we had to resort to heuristic numerical searches to compute the qubit value of the expressions (4). We now present two situations where stronger statements can be made. While techniques have been developed to characterize the boundary of the general quantum set (i.e., with no bound on the dimension) [15], we still lack the efficient tools to characterize the quantum region corresponding to fixed Hilbert-space dimension. The two examples below provide two different approaches to this problem: the first one uses semidefinite programming; the second one establishes a link with the Grothendieck constant.

Using semidefinite programming.—We give here an example of a dimension witness where the maximal violation can be determined for any two-qubit state using semidefinite programming [17]. We consider a scenario where Alice chooses between two settings ($m_A = 2$) and Bob among three settings ($m_B = 3$). All settings yield binary outcomes except Alice's second setting $x = 1$, which is ternary. In this case, the following Bell expression

$$E(\vec{p}) = P_A(0|0) - P(00|00) - P(00|01) - P(00|02) + P(00|10) + P(10|11) + P(20|12) + 1 \quad (5)$$

with local bound $E(\vec{p}) \leq 0$, has recently been introduced [18]. The maximal quantum violation $E^q = 0.2532$ can be found using the method of [15] and is achieved for a partially entangled state of two-qutrits.

In order to prove that the largest violation for qubits is strictly smaller, we computed the maximal value of the right-hand side of (5) over all two-qubit states $\rho \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and over all measurement settings. Since we seek to maximize an expression which is linear in the probabilities $p(ab|xy) = \text{tr}[\rho M_a^x \otimes M_b^y]$, its maximum will be attained by pure states $\rho = |\psi\rangle\langle\psi|$ and extremal POVMs. Up to a local change of basis, any pure two-qubit state can be written as $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$. Every extremal POVM M for qubits has elements $\{M_i\}$ which are proportional to rank 1 projectors [19] and can thus be parameterized in term of the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ as $M_i = \frac{1}{2}(m_i\mathbb{1} + \vec{m}_i \cdot \vec{\sigma})$ where $m_i \geq 0$, $\sum_i m_i = 2$, $\sum_i \vec{m}_i = 0$, and $(m_i)^2 = (\vec{m}_i)^2$. Let u denote the set of variables necessary to represent all POVMs using this parameterizations, and let $c(u) \geq 0$ represent the (quadratic) constraints to which these variables are subject. For given θ , the right-hand side of (5) is a quadratic function $E_\theta(u)$ of u . Our problem is thus to solve the following

(nonconvex) quadratic program

$$E_\theta^* = \max_u E_\theta(u) \quad \text{s.t.} \quad c(u) \geq 0. \quad (6)$$

Solving such a problem is in general a difficult task, as it may have many local optima. Following the approach of Lasserre [16], we derived upper-bounds on E_θ^* using semidefinite programming [17]. For any given value of θ , we obtained an upper-bound on the maximal value of the right-hand side of (5). This value coincides up to numerical precision with the maximum value obtained when we discard one of the outcomes of the POVM $x = 1$. In this case, the inequality (5) reduces to the Clauser-Horne-Shimony-Holt (CHSH) inequality [20], whose maximal violation as a function of θ is $(\sqrt{1 + \sin^2(2\theta)} - 1)/2$. The maximal qubit violation of (5) is thus equal to $E^{(2)} = 1/\sqrt{2} - 1/2 \approx 0.2071$, to be compared with the maximal quantum violation $E^q = 0.2532$ achieved using two-qutrit states. Let us stress that our qubit bound, which can be reached, is an upper bound on the global optimal solution of the problem, since there exist algorithms able to find the global optimum of semidefinite programs [17].

Link to the Grothendieck constant.—The previous examples of qubit witnesses all contain at least one three-outcome measurement. In this case, it is perhaps not surprising, though difficult to prove, that systems of dimension larger than 2 are needed to get the maximal quantum value. In what follows, we show that qubit witnesses exist even for two-outcome correlations, answering Gill's question [8–10] in the bipartite case.

Define the correlator c_{xy} between measurement x by Alice and y by Bob as $c_{xy} = P(a = b|xy) - P(a \neq b|xy)$, and consider now a linear function of such correlators,

$$I = \sum_{i,j=1}^m M_{ij} c_{x_i y_j} \quad (7)$$

defined by an $m \times m$ matrix M verifying the normalization condition $\max_{\{x_i, y_j\}} |\sum_{i,j=1}^m M_{ij} x_i y_j| = 1$ with $x_i, y_j = \pm 1$. Because of this normalization, I can be seen as a standard Bell inequality with local bound 1.

On the other hand, the correlators are quantum, i.e., $c_{xy} = \langle X \otimes Y \rangle_\psi$ for some observables X and Y with ± 1 eigenvalues, if and only if there exist two normalized vectors $\vec{x}, \vec{y} \in R^N$ such that $c_{xy} = \vec{x} \cdot \vec{y}$ (see [21,22] for details). The maximum value that any operator I can take, when c_{xy} is of this form, is known in the mathematical literature as $K_G(N)$ and called the Grothendieck constant of order N ; the maximum over all N is written K_G . Note now that in the case of two-outcome correlators, the analysis can be restricted to projective measurements [23]. In this situation, any Bell operator associated to I is diagonal in the Bell basis, implying that the largest value is obtained for a maximally entangled state, say the singlet. Since any two-outcome correlator for projective measurements on the singlet state is equal to the scalar product of three-

dimensional real vectors, the maximal value of I achievable with qubits is $K_G(3)$. Although the exact values of the Grothendieck constants are still unknown, it is proven that $K_G(3) < K_G$ [24]: this means that there exists an inequality I which is not saturated by correlations coming from two qubits. This proves the existence of dimension witnesses for qubits with two-outcome measurements. Examples of qubit witnesses built from two-outcome measurements were recently found in [9].

We conjecture that two-outcome measurements may be sufficient to test the dimension of any bipartite quantum system, in the sense that there exist dimension witnesses built from binary measurements for any finite dimension. Indeed, all quantum correlators in $\mathbb{C}^d \otimes \mathbb{C}^d$ can be written as a scalar product of vectors of size $2d^2$ [21,22]. Therefore, if $K_G(N)$ is strictly smaller than K_G for any finite N , which is plausible but unproven to our knowledge, one can construct witnesses with binary measurements for arbitrary dimension.

Conclusion and other directions.—With the goal of testing the Hilbert space dimension of an unknown quantum system, we introduced the concept of dimension witness. We presented two examples of qubit witnesses, which can detect correlations that require measurements on quantum systems of dimension greater than two for their generation. Both of these examples involved three-outcome measurements; so the number of measurement outcomes exceeded the dimension of the Hilbert space to be witnessed. This shows, as one may expect, that not all d -outcome correlations can be obtained by measuring quantum systems of dimension smaller than d . Then, somehow surprisingly, we proved that qubit witnesses also exist in the case of two-outcome measurements.

Viewing the dimensionality of a quantum system as a resource and trying to understand how to estimate or bound it is an approach that deserves further investigation. The concept of dimension witnesses represents only a first step in this direction. In general, the problem of Hilbert space characterization is not restricted to a multipartite nonlocal scenario. When dealing with fundamental issues, for instance, it can be relevant to estimate the dimension of a quantum system without any distinction between classical and quantum resources. The motivation is that if nature is indeed described by quantum theory, classical degrees of freedom have also to be coded ultimately in quantum systems. In this context, the (possibly one-partite) global quantum state is then taken to be pure, and the goal is, given some initial statistical data, to determine the physical realization of minimal dimension. One may also wonder to what extent high-dimensional quantum systems are more powerful than lower-dimensional ones when noise is added. In particular, it would be interesting to look for dimension witnesses very robust to noise. Finally, a related, though different question concerns the multipartite case.

How can one be sure that data obtained by measurements on a n -party quantum state do require n -partite entanglement without any assumption of the local Hilbert space dimensions?

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