

Security Bounds for Continuous Variables Quantum Key Distribution

Miguel Navascués and Antonio Acín

ICFO-Institut de Ciències Fotòniques, Jordi Girona 29, Edifici Nexus II, E-08034 Barcelona, Spain

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Security bounds for key distribution protocols using coherent and squeezed states and homodyne measurements are presented. These bounds refer to (i) general attacks and (ii) collective attacks where Eve applies the optimal individual interaction to the sent states, but delays her measurement until the end of the reconciliation process. For the case of a lossy line and coherent states, it is first proven that a secure key distribution is possible up to 1.9 dB of losses. For the second scenario, the security bounds are the same as for the completely incoherent attack.

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Quantum cryptography, that is, quantum key distribution (QKD) followed by one-time pad, allows two honest parties to interchange private information in a completely secure way. Quantum states sent through an insecure channel are used to establish correlations between the sender, Alice, and the receiver, Bob. Since the channel is not secure, the eavesdropper, Eve, can interact with the sent states. However, the no-cloning theorem [1] limits her action: she cannot produce and keep a perfect copy of the intercepted quantum state. After this correlation distribution, Alice and Bob employ reconciliation techniques in order to distill from their list of classical symbols perfectly correlated and completely random bits about which Eve has no information, that is, a secret key. This key is later consumed for sending private information by means of the one-time pad.

The first QKD protocol was introduced by Bennett and Brassard in 1984 [2] and uses two-dimensional quantum systems, or qubits, as information carriers. After it, other QKD protocols were presented [3], using finite-dimensional systems as well. More recently, it has been shown that protocols based on continuous variables' quantum systems could offer an alternative to finite-dimensional schemes, since they may achieve high repetition rates in the near future. The first of these protocols employed squeezed states of light and homodyne measurements [4,5]. Later, a QKD protocol using coherent states and homodyne measurements was proposed in [6] and experimentally demonstrated in [7].

The security of continuous variables' QKD protocols against any type of attack has already been proven, both for the squeezed [5] and the coherent [8] cases. The bounds derived in these works provide sufficient conditions for a secure key distribution. There also exist restricted security proofs (see, for instance, [4,6,9]) where Eve is assumed to apply an incoherent attack; that is, (i) she interacts with the sent states individually and in the same way and (ii) performs incoherent measurements before the reconciliation process has started. The corresponding bounds, then, can be seen as necessary conditions for a secure key distribution [10]. Unfortunately, there exists a clear gap

between the security conditions for general and individual attacks, and we are far from establishing necessary and sufficient conditions for security.

In this work, we analyze the security of QKD protocols employing coherent or squeezed states and homodyne measurements. Using the techniques developed in Refs. [11,12], we find new security bounds for these schemes in the two following scenarios: first, we impose no assumption on Eve's attack and derive a simple condition for general security. Later, we assume that Eve applies to the sent states the optimal individual interaction, but, contrary to previous proofs, she delays her measurement until the end of the reconciliation process. This type of attacks is sometimes called *collective*. In this second scenario, and for the case of a lossy line, we show that the limits for key distillation coincide with those found for incoherent attacks. To our knowledge, this is the first situation in which it is proven that to let Eve delay her measurement until the end of the reconciliation process does not modify the security region.

By completing this work, we learn that similar results have independently been obtained by Grosshans [13].

QKD protocols.—In all the considered protocols, Alice sends to Bob squeezed or coherent states of light modulated by a Gaussian probability distribution. These states propagate through a quantum channel characterized by its transmission T and excess noise ε . Bob randomly measures one of two quadratures, X or P , and communicates the chosen measurement to Alice. Alice and Bob obtain a list of correlated real numbers from which the key has to be extracted. There exists an entanglement-based protocol that is completely equivalent to this prepare and measure (PM) scheme [14]. Indeed, Alice's preparation can be done by measuring half of a two-mode squeezed state, of squeezing parameter r_A , as shown in Fig. 1. For instance, Alice can measure both quadratures, X_A and P_A , after the beam splitter of transmittivity $T_A = 1/2$. The corresponding PM scheme consists of Alice sending coherent states with Gaussian probability distributions of variance $\langle X^2 \rangle = \langle P^2 \rangle = (\cosh(r) - 1)/2$. On the other hand, if $T_A = 1$ and Alice chooses randomly the measured quadrature, she is

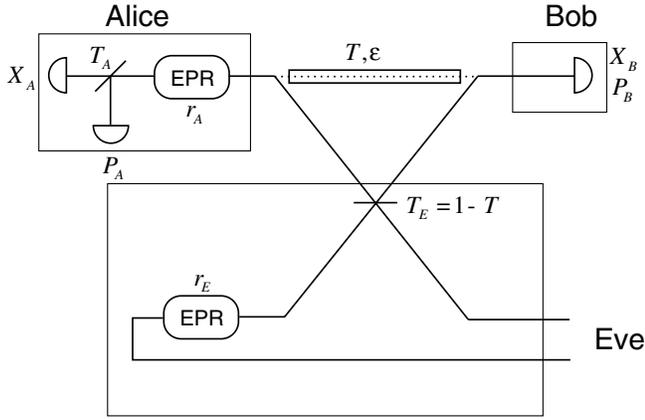


FIG. 1. The picture shows the considered protocols. After Alice's effective preparation using an entangled state, coherent or squeezed states of light are sent to Bob, according to a Gaussian modulation. Eve replaces the channel by the entangling cloner of parameters r_E and T_E .

effectively sending squeezed states of squeezing parameter $\cosh(r)$ and modulated with a Gaussian distribution of variance $\langle X^2 \rangle = \sinh(r)^2 / (2 \cosh(r))$. This entanglement description simplifies the theoretical analysis of the protocols, but the obtained security bounds automatically apply to the corresponding PM scheme.

It is also convenient at this point to introduce Eve's optimal individual attack, the so-called entangling cloner. As proven in [9,15], the optimal way in which Eve can "simulate" the channel (T, ε) is by combining into a beam splitter of transmittivity $T_E = 1 - T$, the intercepted state and half of a two-mode squeezed state. The squeezing parameter, r_E , has to be chosen such that $(1 - T) \cosh r_E = 1 - T + \varepsilon T$.

General security proof.—Recently, powerful techniques for the analysis of general security proofs of any QKD protocol have been presented in [11]. In any QKD scheme, there is a tomographic process that partly characterizes the insecure channel connecting Alice and Bob. It allows the honest parties to evaluate their mutual information, I_{AB} . Moreover, it puts a bound on Eve's knowledge: it has been shown in [11] that, using the information collected during this process, one can construct a secure reconciliation protocol that allows one to extract

$$K = I_{AB} - \max_{\rho_{AB} \in \mathcal{R}} S(\rho_{AB}), \quad (1)$$

secret bits, where \mathcal{R} is the set of quantum states consistent with the measured probabilities (see [11] for more details). Thus, this quantity represents a lower bound to the achievable key rate, $K_{\text{opt}} \geq K$. For continuous variable systems, if Alice and Bob monitor the channel by means of the first and second moments of their data, the state ρ_{AB} of maximal entropy in (1) has to be Gaussian. A proof of this result can be found in [16].

A simple calculation shows that for the same measured quadrature, the joint probability distribution of Alice and Bob's results is Gaussian with covariance matrix

$$\gamma_{AB} = \begin{pmatrix} T_A \cosh r_A + R_A & \sqrt{T_A T} \sinh r_A \\ \sqrt{T_A T} \sinh r_A & T \cosh r_A + R \cosh r_E \end{pmatrix}, \quad (2)$$

where $R = 1 - T$ and the same for R_A . This gives the first term in (1). The second term can be computed from ρ_E , since $S(\rho_{AB}) = S(\rho_E)$. Eve's state is a two-mode Gaussian state, with covariance matrix $\gamma_E = \gamma'_E \mathbb{1}_2$,

$$\gamma'_E = \begin{pmatrix} T \cosh r_E + R \cosh r_A & 0 \\ 0 & \cosh r_E \end{pmatrix}. \quad (3)$$

Using (2) and (3), it is straightforward to compute K as a function of T_A, r_A, T, ε .

For the case of a lossy line, $\varepsilon = 0$, one can numerically see that there exists an optimal squeezing r_A^{opt} for both the coherent and squeezed case (see Fig. 2). A possible reason for this counterintuitive result may be that K is known to be a nontight bound to the optimal key rate [11]. This optimal squeezing is the same for squeezed and coherent states, $r_A^{\text{opt}} \approx 1.5$, and defines a critical value for the tolerable losses of approximately 1.7 and 0.83 dB.

As discussed in [11] it is possible to improve the bound (1) by conditioning the privacy amplification process on a classical random variable W (see [11] for more details), decreasing Eve's entropy. For the case of coherent states, Alice and Bob can make public the value of the second measured quadrature, instead of discarding it [13]. This process does not modify Alice and Bob's mutual information but changes Eve's entropy. The obtained critical transmission, T_c , is now a decreasing function of the squeezing, as expected. One can see that in the limit of high modulation, $r_A \rightarrow \infty$,

$$T_c = \frac{e^2}{e^2 + 4}. \quad (4)$$

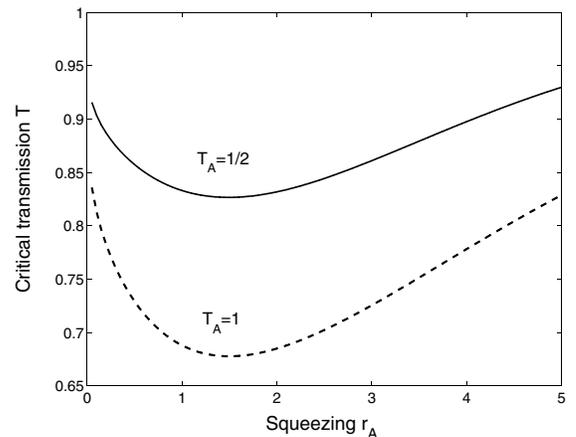


FIG. 2. Tolerable losses as a function of the modulation for protocols using coherent (solid line) and squeezed (dashed line) states. The optimal value for both schemes is $r_A \approx 1.5$.

That is, the protocol using coherent states and homodyne measurements is secure up to 1.9 dB of losses.

Collective attack.—As noted above, the bound (1) is very useful because it does not make any assumption on Eve’s attack, but is known not to be tight. Moreover, it does not allow one to distinguish between direct and reverse reconciliation protocols, where the (one-way) flow of information goes from Alice to Bob and vice versa. This distinction does not play any role in QKD protocol using finite-dimensional quantum systems, but is relevant for continuous variables protocols [9]. Indeed, for the case of a lossy line and incoherent attacks, the value of the channel transmission limiting the security is equal to 1/2 for direct reconciliation, while it goes to zero for reverse reconciliation, for squeezed and coherent states [9].

In what follows, we consider an attack where the only assumption is that Eve applies to the intercepted states the optimal individual interaction. In the entanglement picture, this corresponds to the case where Alice, Bob, and Eve share N independent realizations of a quantum state $|\Psi_{ABE}\rangle$. This scenario is again represented in Fig. 1. However, contrary to the usual incoherent attacks previously studied, Eve delays her measurement until the end of the reconciliation protocol. Note that this attack is clearly coherent, because Eve can globally measure her N quantum states. Moreover, she can optimize her measurement according to all the communication interchanged during the whole reconciliation process.

After Alice and Bob’s measurements, the three parties share N independent realizations of classical-classical-quantum correlated variables A , B , and $|\psi_E\rangle$. Under the N independent realizations assumption, it has been shown in [12] that there exists a key distillation protocol achieving a rate

$$K' = I_{AB} - \chi(A:E), \quad (5)$$

where χ denotes the Holevo bound [17]. This security condition is rather intuitive: if this quantity is positive, the information Bob has on Alice’s symbols is larger than the classical information accessible to Eve through the quantum channel Alice-Eve. The results of [12] prove that this advantage can, indeed, be exploited for distilling a key. When considering reverse reconciliation, a similar expression holds where $\chi(A:E)$ is replaced by $\chi(B:E)$.

For a lossy line and direct reconciliation, the critical transmission, T_c , limiting the security is again a decreasing function of the squeezing r_A . In the limit of very high modulation one can see that T_c is the solution to the equation

$$\frac{T_c(1 - T_c)[1 - T_c + T_A(2T_c - 1)]}{T_A + T_c - 2T_AT_c} = (1 - T_c)^2. \quad (6)$$

Remarkably, $T_c = 1/2$ is the searched solution $\forall T_A$; i.e., we recover the same value as for the completely incoherent attack [6]. Therefore, by increasing the modulation r_A , the

limiting losses’ value for a key distribution secure against the considered collective attack tends to 3 dB.

A stronger result is obtained for reverse reconciliation. In the limit of high losses, $T \ll 1$, the key rate (5) reads

$$K' \approx \frac{\cosh r_A - 1}{4} \left[\frac{2(\cosh r_A + 1)T_A}{1 + T_A(\cosh r_A - 1)} + (\cosh r_A - 1) \log \left(\frac{\cosh r_A - 1}{\cosh r_A + 1} \right) \right], \quad (7)$$

which is always positive. Thus, there is no loss limit for reverse reconciliation protocols. But perhaps more surprisingly, there is no need of high modulation or squeezing for recovering the same limits as for the completely incoherent attack. That is, a protocol using coherent states and any modulation is secure for all line transmissions, even if Eve is assumed to delay her measurement until the end of the whole reconciliation process.

Concerning the amount of excess noise the protocols tolerate, this is shown in Fig. 3. For squeezed states, it is always more convenient to employ reverse reconciliation techniques. For the case of coherent states, direct reconciliation turns out to be more resistant against excess noise up to a channel transmission of ≈ 0.65 . Note that there exist limiting values of the excess noise, ε_c , for which the considered key rates are zero, independently of the modulation and the losses. These values can be computed analytically. For coherent states and direct reconciliation, one has that ε_c is the solution to the equation

$$\frac{1}{1 + \varepsilon} \left(\frac{\sqrt{1 + \varepsilon} + 1}{\sqrt{1 + \varepsilon} - 1} \right)^{\sqrt{1 + \varepsilon}} = e^2 \quad (8)$$

that gives $\varepsilon_c \approx 0.8$, while for reverse reconciliation

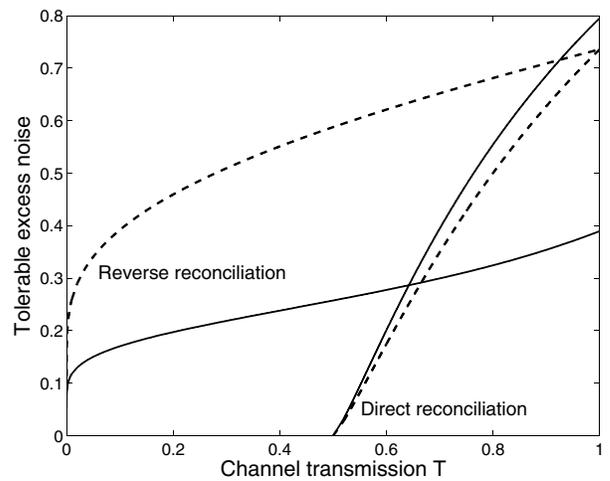


FIG. 3. Tolerable excess noise as a function of the losses for reverse and direct reconciliation and squeezed (dashed line) and coherent (solid line) states. All the curves have been computed in the limit of very high modulation, $r_A \rightarrow \infty$.

$$\varepsilon_c = \frac{1}{2} \left(\sqrt{1 + \frac{16}{e^2}} - 1 \right) \approx 0.39. \quad (9)$$

In the case of squeezed states, the critical noise is equal to $2/e \approx 0.7$ for both reconciliation protocols.

Concluding remarks.—In this work we have applied the recent security proofs of Refs. [11,12] to QKD protocols using coherent and squeezed states and homodyne measurement. It has to be clear that the obtained results provide lower bounds to the achievable secret key rate. Thus, they represent sufficient conditions for key distillation for the studied scenario (and assumptions).

The first of the analyzed conditions is very powerful because it does not make any assumption on the eavesdropping attack. For a lossy line and coherent states, it has been shown here that a secure key distribution is possible up to 1.9 dB of losses. Existing proofs of security work up to 1.4 [8] and 1.6 dB [5] of losses. Thus, our results slightly improve the known region of general security, without requiring any squeezing.

The second type of bounds does not refer to the most general situation, since Alice, Bob, and Eve are assumed to share N linearly independent copies of a quantum state $|\Psi_{ABE}\rangle$. We have considered the case where Eve applies the optimal individual interaction [14,15] to any sent state. Therefore, in this first step of her attack, Eve is assumed to introduce no correlations among the quantum states shared by Alice and Bob. The bounds obtained in this scenario cannot be seen as proofs of general security. However, it is now possible to distinguish between direct and reverse reconciliation, a relevant issue for continuous variables quantum cryptography. Remarkably, the obtained bounds turn out to be the same as for the fully incoherent attack. The case of reverse reconciliation is perhaps more surprising, since this is true for any value of Alice's modulation. As far as we know, this is the first situation in which it is proven that allowing Eve to delay her measurement does not give her any significant advantage (see also [13,18]).

We conclude with a brief comment on the N independent copies assumption required for the bound (5). Our results suggest that the only possibility left to Eve that could modify the security bounds for the fully incoherent attack would be to introduce correlations among the different copies of the states. Does this fact provide any improvement on her attack? As discussed in what follows, one could expect this not to be the case. After some channel tomography, Alice and Bob know to share a state $\rho_{AB}^{(N)}$ such that any single copy is a state $\rho_{AB} \in \mathcal{R}$, consistent with their measured probabilities. They should assume that Eve has tried to be as correlated as possible to their state. Since the global state is pure, this means that Eve has optimized the entanglement of $|\Psi_{ABE}\rangle$ over the splitting $AB - E$. Thus, she has maximized the entropy of entanglement [19] of $|\Psi_{ABE}\rangle$, i.e., the entropy of the local state $\rho_{AB}^{(N)}$,

subject to the constraint that the single-copy state is ρ_{AB} . This maximization naturally leads to the N independent copies assumptions, since $S(\rho_{AB}^{(N)}) \leq NS(\rho_{AB}) = S(\rho_{AB}^{\otimes N})$. Although far from being a proof, this simple argument, as well as Eq. (1), suggests that the best attack could consist of Eve preparing N copies of the most entropic state. If this was true, the bounds derived from (5) would hold and provide a necessary and sufficient condition for a secure QKD over a lossy line using coherent (or squeezed) states and homodyne measurements.

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